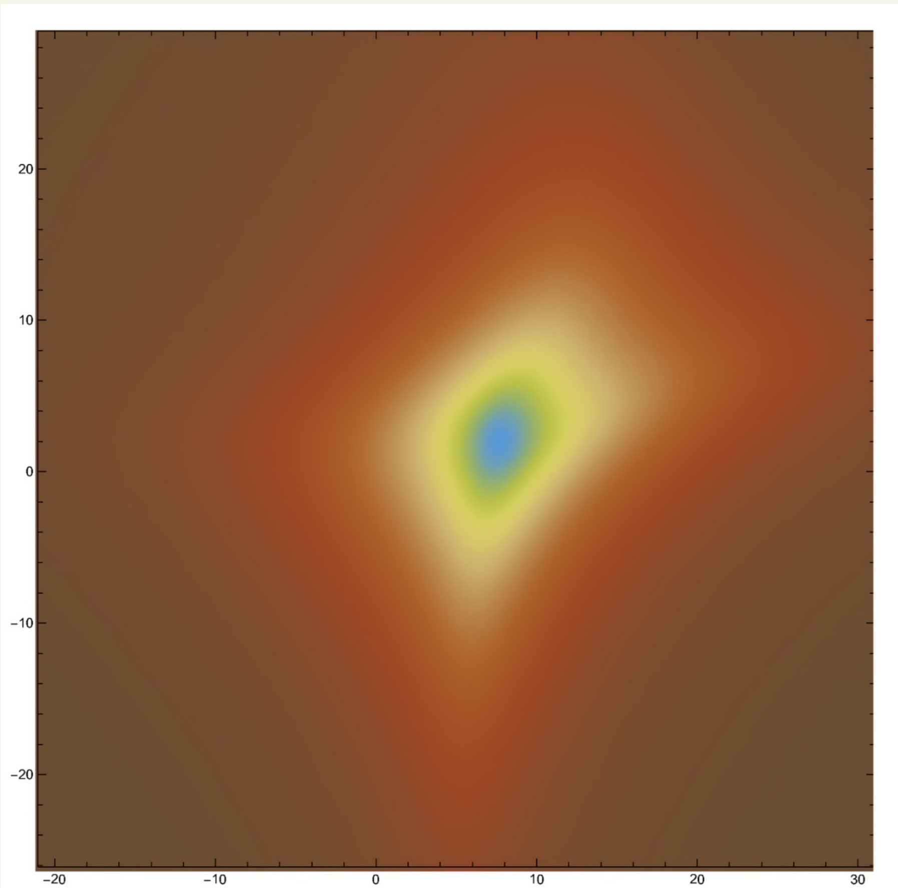
Scattering Amplitudes of Stable Curves



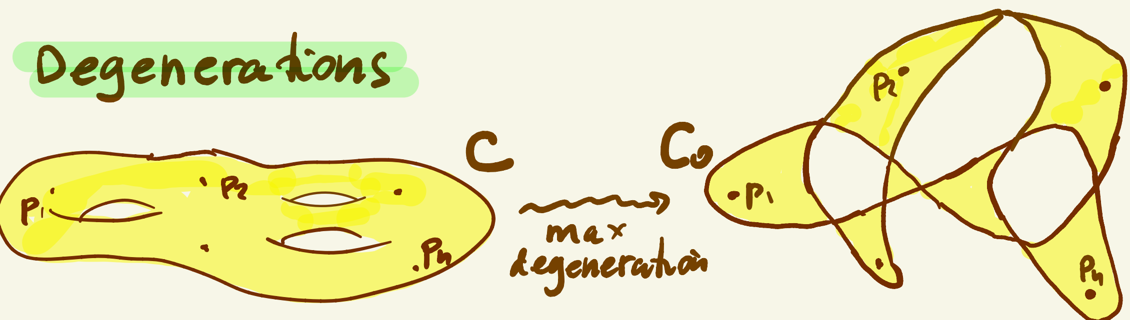
Jenia Tevelev (UMass Amherst)

Probabilistic Brill-Noether theory

What can we say about the distribution of $(q_1 \dots q_n)$?

Def MHV case: $d = g+1$, $n = g+3$. $\varphi = \varphi_L$, where $L \in \mathrm{Pic}^d C = \mathbb{C}^g/\Lambda$ uniformly distributed w.r.t. $dz_1 \wedge \dots \wedge dz_g$. Scattering amplitude is the probability distribution of $(q_1, \dots, q_n) \in M_{0,n}$

Degenerations

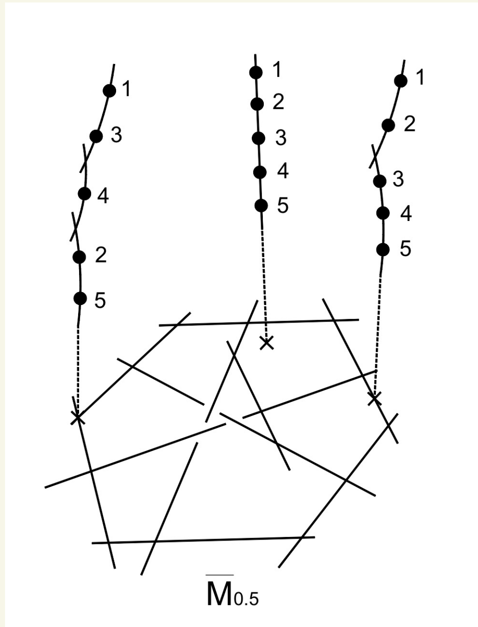


scattering amplitude of a smooth curve "⇝" leading singularity of a MHV scattering amplitude form of n particles in $\mathcal{N}=4$ SYM obtained by putting some internal particles on-shell (Arkani-Hamed, Bourjaily, Cachazo, Postnikov, Trnka) [ABC+]

degenerate & choose branch

Origins of the question

what is $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$?

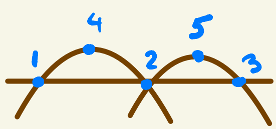
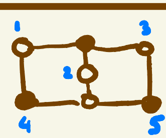
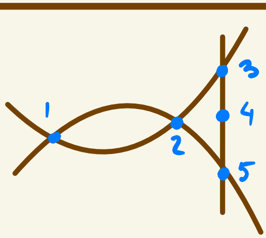
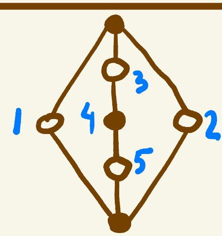


- Every boundary divisor is contracted by a birational morphism (to $\mathbb{P}^{n-3}$) $\Rightarrow$ it generates an extremal ray

- (Keel, Vermeire) new divisors are necessary already for $\overline{M}_{0,6}$

- (Castravet - T) A hypertree is a collection of triples $\Gamma_1, \dots, \Gamma_d \subset \{1 \dots n\}$ such that

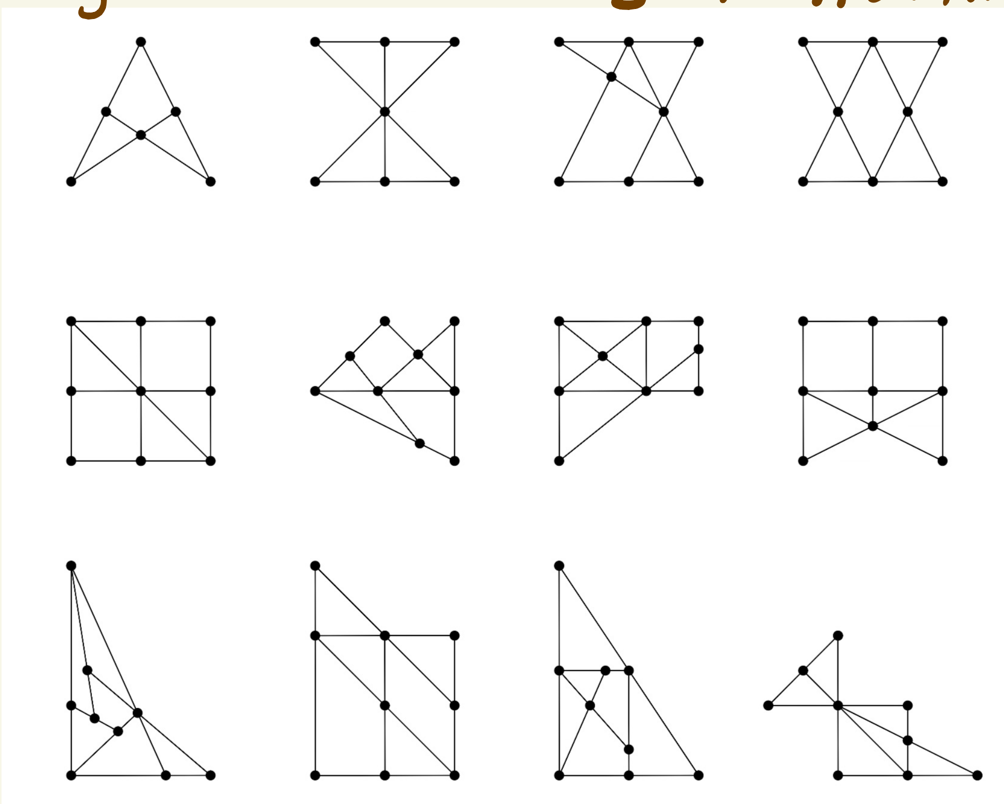
$$\Big|\bigcup_{j\in S} \Gamma_j\Big| \geq |S|+2 \quad \text{for every } S \subset \{1 \dots d\}$$

triples	sticky curve	on-shell diagram
123 124 235		
123 125 345		

Th (Castravet-T, Crelle 2013)

If a hypertree is irreducible (i.e. $|\bigcup_{j\in S} T_j| > |S|+2$ unless $|S|=1$ or $d-2$) then it gives an effective divisor on $\overline{M}_{0,n}$ contractible by a "Brill-Noether"-type birational contraction,

$\Rightarrow$ it gives an extremal ray of $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$.



But... $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ has more generators!

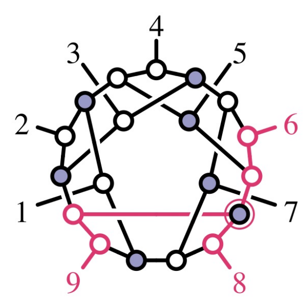
(Opie, Doran-Giansiracusa-Jensen, ...)

Th (Castravet-Laface-T-Ugaglia, 2020)

$\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ is in finitely generated for $n \geq 10$,

what kind of geometric objects on $M_{0,n}$ correspond to arbitrary hypertrees?

Leading singularities of scattering amplitudes!



$$\Leftrightarrow \begin{Bmatrix} (124) \\ (189) \\ (293) \\ (364) \\ (465) \\ (687) \\ (689) \end{Bmatrix}$$

[ABC+]

$$\frac{\left(\langle 91\rangle\langle 32\rangle\langle 46\rangle - \langle 16\rangle\langle 43\rangle\langle 29\rangle\right)^2}{\langle 12\rangle\langle 24\rangle\langle 41\rangle\langle 18\rangle\langle 91\rangle\langle 29\rangle\langle 93\rangle\langle 32\rangle\langle 36\rangle\langle 43\rangle\langle 65\rangle\langle 54\rangle\langle 87\rangle\langle 76\rangle\langle 69\rangle}.$$

monomial Parke-Taylor amplitudes, ...

classical BN theory	$\rightsquigarrow$	irreducible hypertree divisors
probabilistic BN theory	$\rightsquigarrow$	leading singularities of MHV scattering amplitudes

MHV case: $d=g+1,\ n=g+3$.

Lemma-Definition

$\Lambda : \mathrm{Pic}^d C \dashrightarrow M_{0,n}\ ,\ L \mapsto (\varphi_L(p_1), \dots, \varphi_L(p_n))$ is a generically finite scattering amplitude map

$A = dz_1 \wedge \dots \wedge dz_g$ (viewed as a multivalued meromorphic top degree form on $M_{0,n}$) is a scattering amplitude form

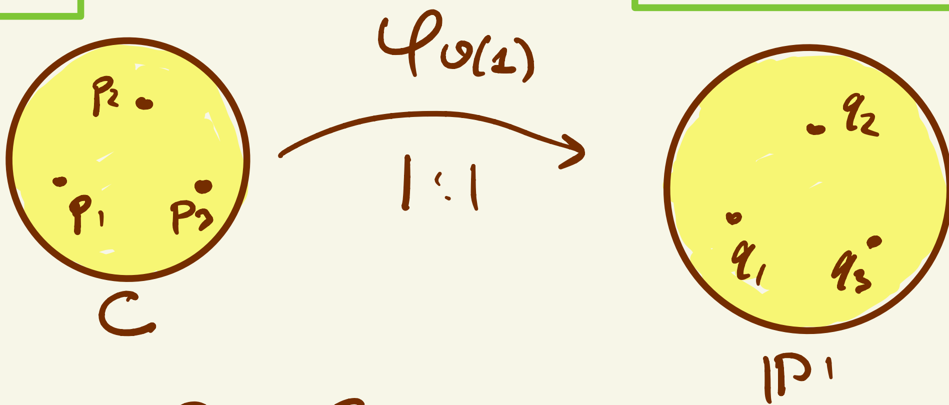
Theorem Λ has degree 2^g for a general $(C; p_1 \dots p_n) \in M_{g,n}$

Goal: "Separate" 2^g branches of Λ (to have single-valued probability measures)

I will give several interpretations of 2^g.

Another approach is due to Cela-Pandharipande-Schmitt & Farkas-Lian

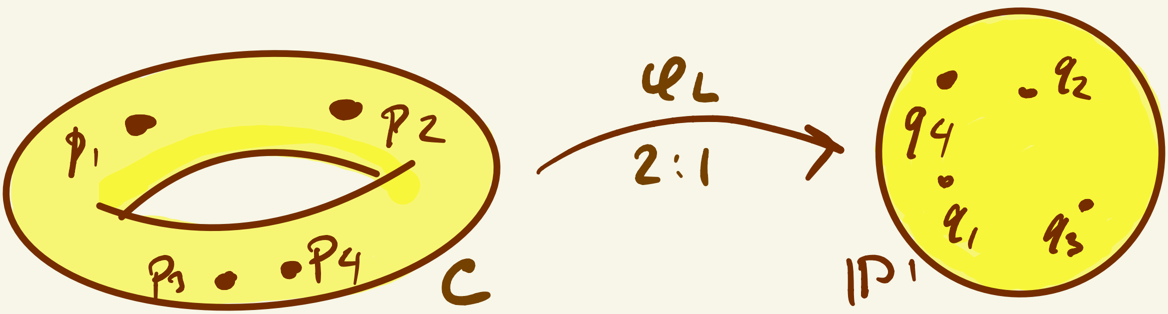
$\boxed{g=0}$ $\boxed{d=1,\ n=3}$



$$\Lambda : \mathrm{Pic}^1 C \longrightarrow M_{0,3}$$

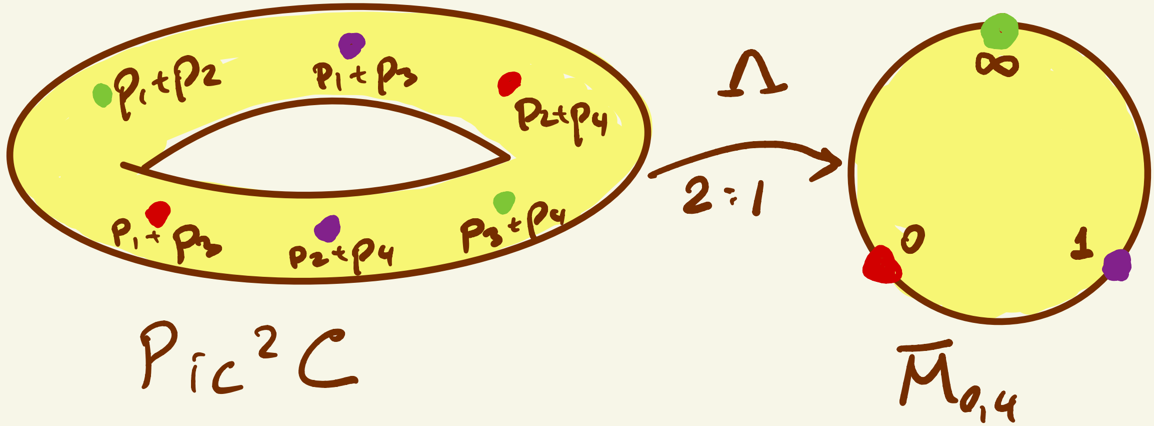
identity map

$\boxed{g=1}$ $\boxed{d=2,\ n=4}$



$$\Lambda : \mathrm{Pic}^2 C \longrightarrow M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$$

$L \longmapsto$ cross-ratio of $q_1, \dots, q_4 \in \mathbb{P}^1$



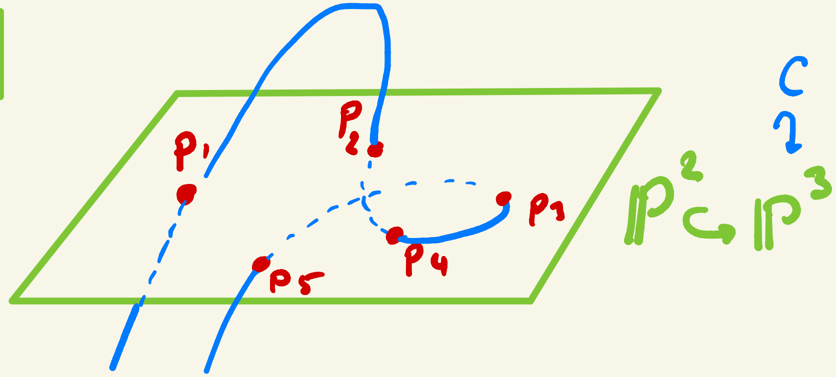
$A = dz = \dfrac{dx}{\sqrt{f_4(x)}}$ is an integrand of an elliptic integral

A recovers $(C; p_1 \ldots p_4) \in M_{1,4}$ modulo the action of the Klein 4-group on $\{p_1 \ldots p_4\}$

$\boxed{g=2}$ (1) For hyperelliptic curves ($g \geq 2$), we assume (following Mumford) that $p_i \neq \bar{p}_j$ for $i \neq j$.

(2) [Halphen] $\varphi_{p_1+\ldots+p_n} : C \hookrightarrow \mathbb{P}^3$ (for general $p_1 \ldots p_n$)

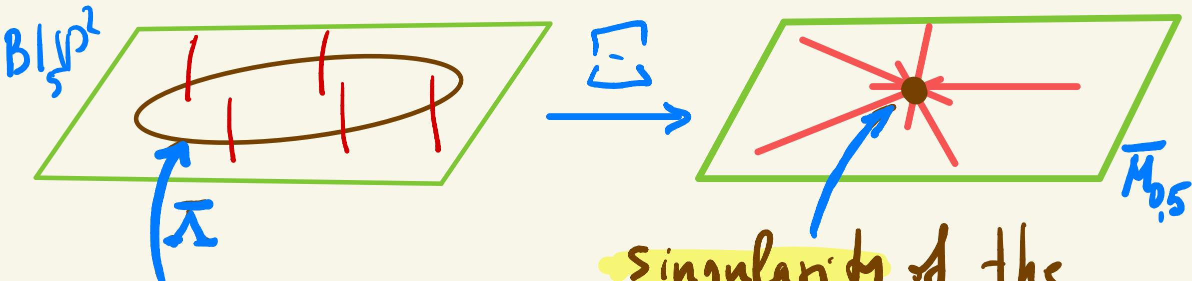
$\boxed{g=2}$



$$\mathrm{Sym}^2 C \overset{4:1}{\dashrightarrow} \mathbb{P}^2 \qquad (p,q) \longmapsto \mathbb{P}^2 \cap \ell_{pq}$$

$$\begin{array}{ccc}
\mathrm{Bl}_{15}\,\mathrm{Sym}^2 C & \xrightarrow[\overline{\Lambda}]{4:1} & \mathrm{Bl}_5\,\mathbb{P}^2 \\
\| & \text{finite} & \downarrow \Xi \\
\mathrm{Bl}_{16}\,\mathrm{Pic}^3 C & \xrightarrow[4:1]{\Lambda} & \overline{M}_{0,5} \quad (\simeq \mathrm{Bl}_4 \mathbb{P}^2) \\
 & \text{gen. finite} &
\end{array}$$

$$(p,q) \longmapsto \mathcal{O}(\Sigma p_i - p - q)$$



$K_C + C \subset \mathrm{Pic}^3 C$

singularity of the scattering amplitude form

$$\mathrm{Bl}_{16}\,\mathrm{Pic}^3 C \xrightarrow[4:1]{\overline{\Lambda}} \mathrm{Bl}_5\,\mathbb{P}^2$$

16 (-1)-curves
16 Θ-divisors $\longrightarrow$ 16 (-1)-curves

Classical Case: $p_1 \cdots p_5 \in C$ are **Weierstrass points**. Then $\overline{\Lambda}$ factors

$$\overline{\Lambda} : \mathrm{Bl}_{16}\,\mathrm{Pic}^3 C \xrightarrow{2:1} \mathrm{K3} \xrightarrow{2:1} \mathrm{Bl}_5\,\mathbb{P}^2$$

minimal resolution of the Kümmer surface

when marked points $p_1 \cdots p_5 \in C$ move away from the weierstrass points, the K3 disappears but the 4:1 cover survives!

Hyperelliptic curves $C = \{y^2 = f(z)\}$

$$\varphi_h : C \xrightarrow{2:1} \mathbb{P}^1 \quad \varphi_h(p_i) = z_i \;, \quad z_i \neq z_j$$

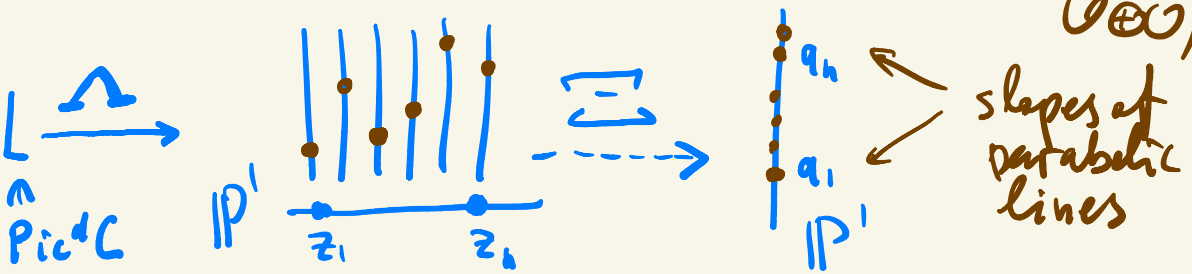
$$
\begin{tikzcd}
\operatorname{Pic}^d C \arrow[r, "\bar{\Lambda}"] \arrow[dr, dashed, "\Lambda"'] & \operatorname{Bun}(\mathbb{P}^1; z_1 \dots z_n) \arrow[d, dashed, "\Xi\ \text{(birational)}"] \\
& M_{0,n}
\end{tikzcd}
$$

Bun is a stack of quasi-parabolic vector bundles on $\mathbb{P}^1$ of rank 2, determinant 0.

$\bar{\Lambda} : L \longmapsto E = (\varphi_n)_* L$

parabolic lines $V_i = \operatorname{Ker}[E_{z_i} \to L_{p_i}]$

Generically, $\mathbb{P}(E) = \mathbb{P}^1 \times \mathbb{P}^1$ (splitting type $\mathcal{O} \oplus \mathcal{O}$)



Singularity in the scattering amplitude form:

$$
\begin{tikzcd}
\theta\text{-divisor} \subset \operatorname{Pic}^d C \arrow[r, "\bar{\Lambda}"] \arrow[dr, dashed, "\Lambda"'] & \operatorname{Bun} \supset \text{splitting type } \mathcal{O}(-1) \oplus \mathcal{O}(1) \arrow[d, dashed, "\Xi"] \\
& M_{0,n} \ni (z_1, \dots, z_n)
\end{tikzcd}
$$

Matrix Model

$$\mathrm{Pic}^d C - \Theta = \left\{ \underbrace{\begin{bmatrix} v(z) & u(z) \\ w(z) & -v(z) \end{bmatrix}}_{} : \begin{array}{l} \deg u, v, w \le d \\ v^2 + uw = f(z) \end{array} \right\} \Big/ PGL_2$$

$M(z)$

Λ

Slope of one of the eigenspaces of $M(z_i)$

$$M_{0,n} \subset (\mathbb{P}^1)^n / PGL_2$$

$$\deg \Lambda = \frac{\left| \{M(z)\} \cap \left\{ \begin{array}{l} \text{matrices with fixed} \\ \text{eigenspaces at } z_1 \ldots z_n \end{array} \right\} \right|}{2^n}$$

$$= 2^{2g+3} \Big/ 2^n$$

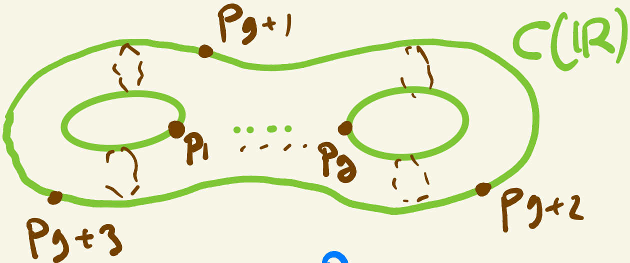
# of coefficients of $f(z)$

$$= 2^g$$

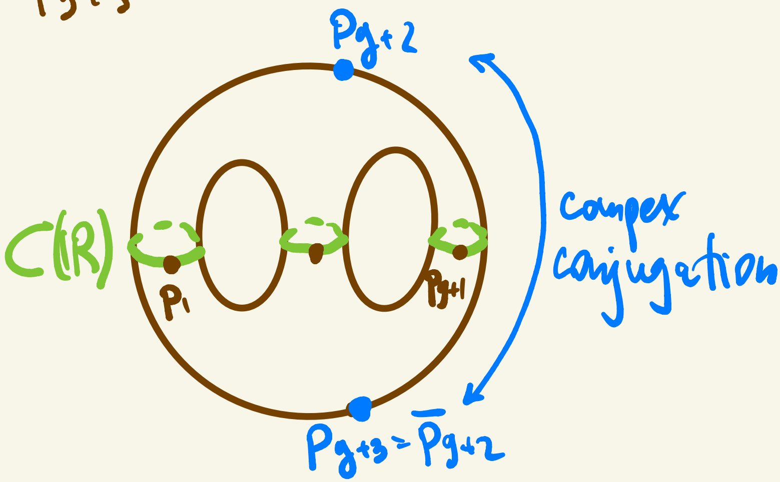
Amplification. Translation-invariant vector fields on $\mathrm{Pic}^d C$ are given by the Lax pair differential equations in the matrix model. Multivalued form A on $M_{0,n}$ is obtained by wedging these fields, dualizing and expressing in terms of slopes of eigenspaces

Scattering on M-curves

$C(\mathbb{R}) = C_1 \cup \dots \cup C_{g+1}$



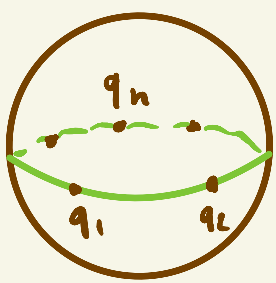
(Type A)



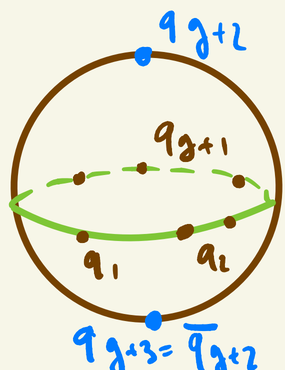
(Type B)

$$\Lambda\left(\mathrm{Pic}^d(C)(\mathbb{R})\right) \hookrightarrow M^{A/B}_{0,n}(\mathbb{R})$$

real form :



$\mathbb{P}^1(\mathbb{R})$



(Type A) (Type B)

$M^A_{0,4}(\mathbb{R}) = \mathbb{R} - \{0, 1, \infty\}$ $M^B_{0,4}(\mathbb{R}) = \{\mathrm{Re}\, z = \frac{1}{2}\}$

For an M-curve C, $\mathrm{Pic}^d(C)(\mathbb{R})$ has 2^g connected components indexed by subsets $I \subset \{1 \dots g+1\}$ such that $|I| \equiv d \mod 2$

$L \in \mathrm{Pic}_I \iff L = \mathcal{O}(D)$ where $D = \overline{D}$ and $\deg(D \cap C_i) \equiv I \cap \{i\} \mod 2$

Each Pic_I is a torsor over $\mathrm{Pic}_{\emptyset} \simeq (\mathbb{R}/\mathbb{Z})^g$ A/B

Main Theorem

The scattering amplitude $\Lambda : \mathrm{Pic}^d(C) \dashrightarrow M_{0,n}$ satisfies (1) $\Lambda^{-1}(M_{0,n}^{A/B}(\mathbb{R})) \subset \mathrm{Pic}^d(C)(\mathbb{R})$

(2) Λ induces a real-analytic isomorphism

$$(\mathbb{R}/\mathbb{Z})^g \simeq \mathrm{Pic}_I \supset \underset{\text{Zariski open}}{\mathcal{U}_I} \xrightarrow{\Lambda} \underset{\text{Zariski open}}{V_I} \subset M_{0,n}^{A/B}(\mathbb{R})$$

Corollary $\Lambda : \mathrm{Pic}^d(C) \dashrightarrow M_{0,n}$ has degree 2^g for a general $C \in M_{g,n}$

Pf The locus of M-curves of type A (or B) is Zariski-dense in $M_{g,n}$ and $M_{0,n}^{A/B}(\mathbb{R})$ is Zariski-dense in $M_{0,n}$ ∎

Amplification

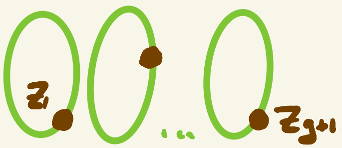
Volume forms on $\mathrm{Pic}_I \cong (\mathbb{R}/\mathbb{Z})^g$ give 2^g single-valued probability distributions on $M_{0,n}^{A/B}(\mathbb{R})$

Amplification

$\mathrm{Pic}_H := \mathrm{Pic}_{\{1 \dots g+1\}}$ is called the Huisman component

$D \in \mathrm{Pic}_H$ effective

$\iff D = z_1 + \dots + z_{g+1}$



$z_i \in C_i$

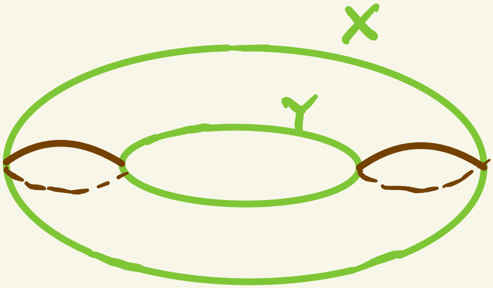
Theorem Compactify $M_{0,n}$ by $(\mathbb{P}^1)^g$ using cross-ratios

$\pi_{i,g+1,g+2,g+3} : M_{0,n} \to M_{0,4}$ $i = 1 \dots g$.

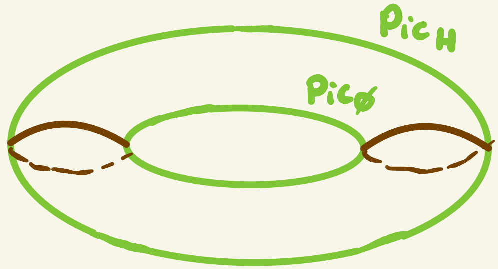
Then Λ induces a real analytic isomorphism $\mathrm{Pic}_H \to (\mathbb{RP}^1)^g$

The corresponding probability density is smooth and positive

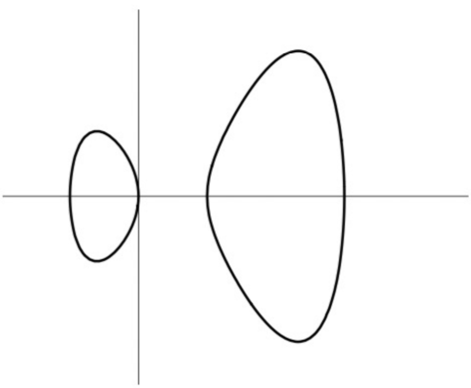
$\boxed{g=1}$



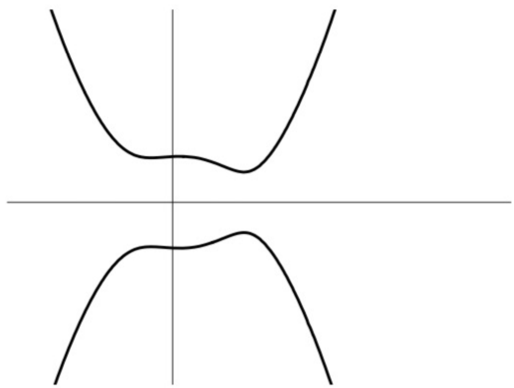
C



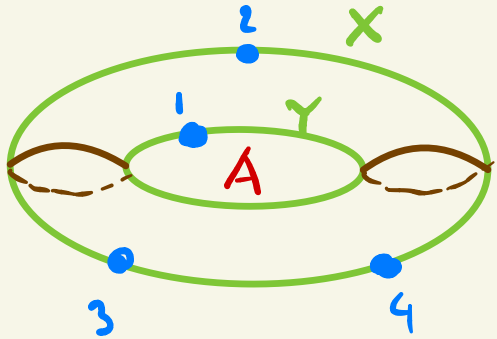
$\mathrm{Pic}^2 C$



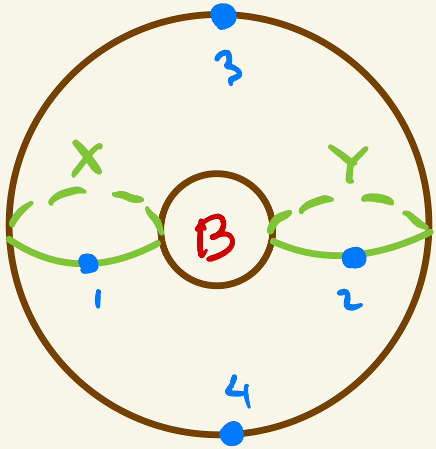
$L \in \mathrm{Pic}_\emptyset$



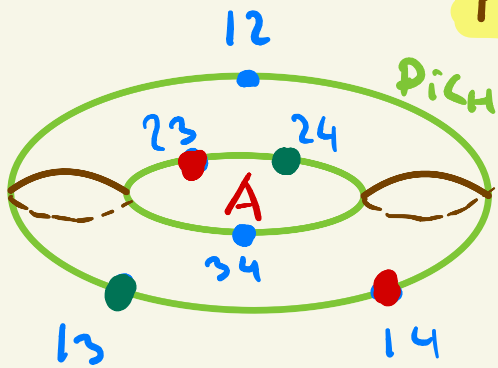
$L \in \mathrm{Pic}_H$

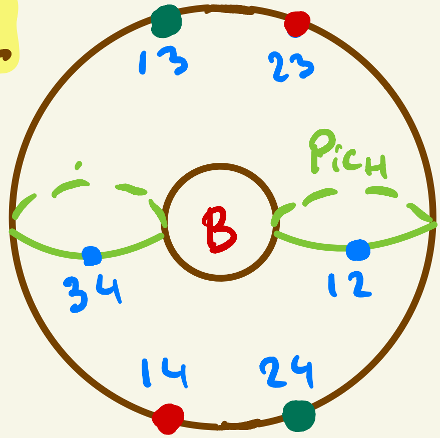


C



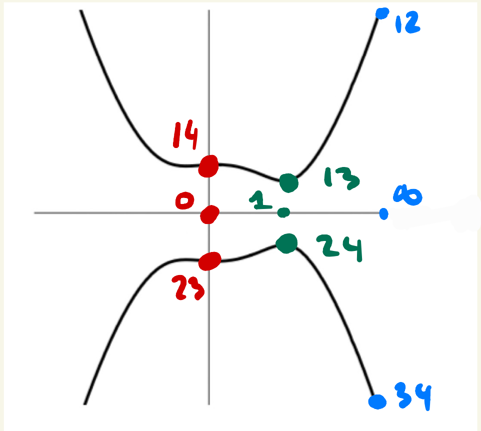
$\mathrm{Pic}^2 C$



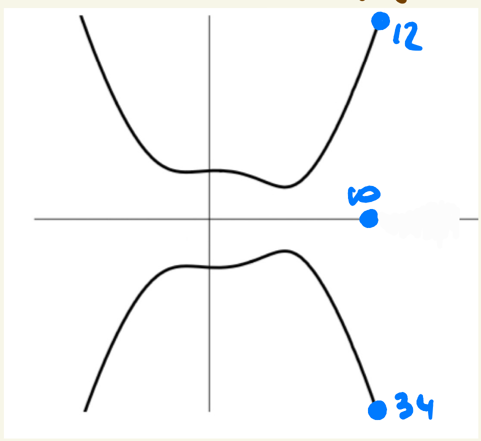


$\Lambda : \mathrm{Pic}^2 C \xrightarrow{2:1} \overline{M}_{0,4} = \mathbb{P}^1$ is given by

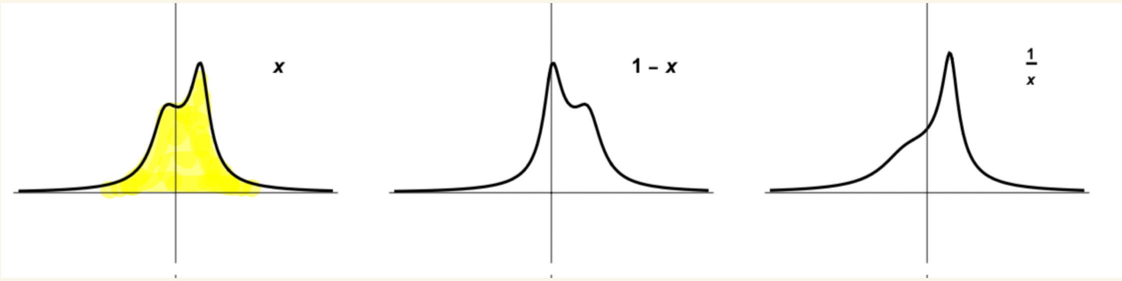
$$\mathcal{O}(p_{12}+p_{34}) \simeq \mathcal{O}(p_{13}+p_{24}) \simeq \mathcal{O}(p_{23}+p_{14}) \in \mathrm{Pic}_H(\mathrm{Pic}^2 C)$$



Λ

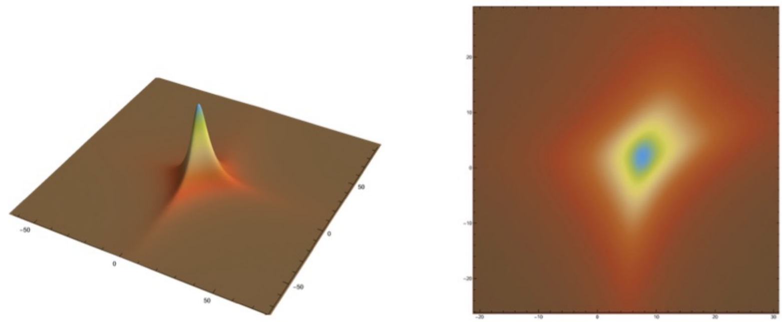


Genus 1 scattering measure

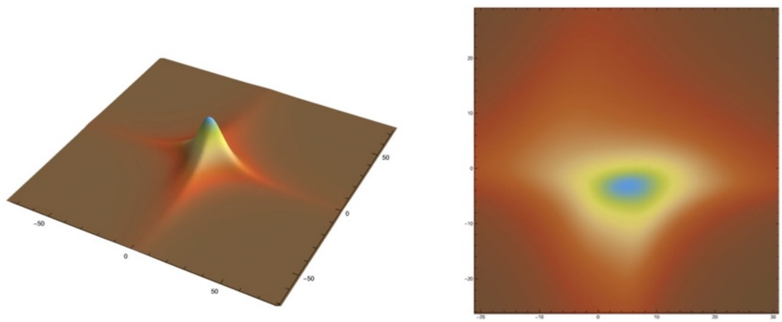


Genus 2, Huisman's Component

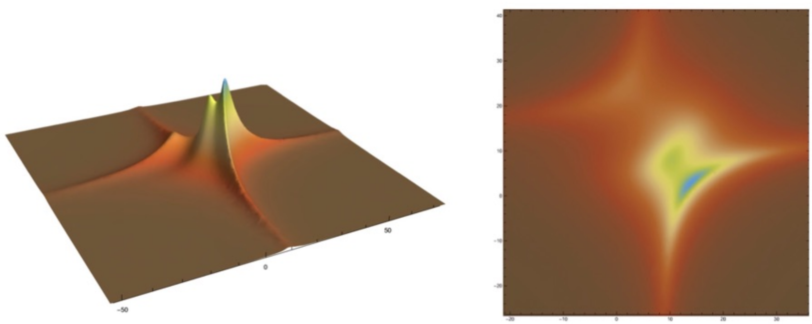
Roots: -1, 0, 1, 2, 3
Marked points:
(-0.7, 1.8885)
(1.3, 1.03317)
(3.3, 3.56769)
(3.45, 4.95408)
(4.2, -13.5832)



Roots: -1, 0, 1, 2, 3
Marked points:
(-0.1, 0.802801)
(1.9, 0.738573)
(3.9, 9.73481)
(4.35, 15.727)
(6.6, -68.2029)



Roots: -2, -0.2, 0.2, 2, 7
Marked points:
(-1.28, 5.59063)
(0.92, 3.93215)
(7.4, 33.3322)
(7.6, 43.1488)
(8.6, -90.9632)



Planar locus $W = \{L : h^0(L) > 2\}$

has codimension 3 in $\mathrm{Pic}^d C$

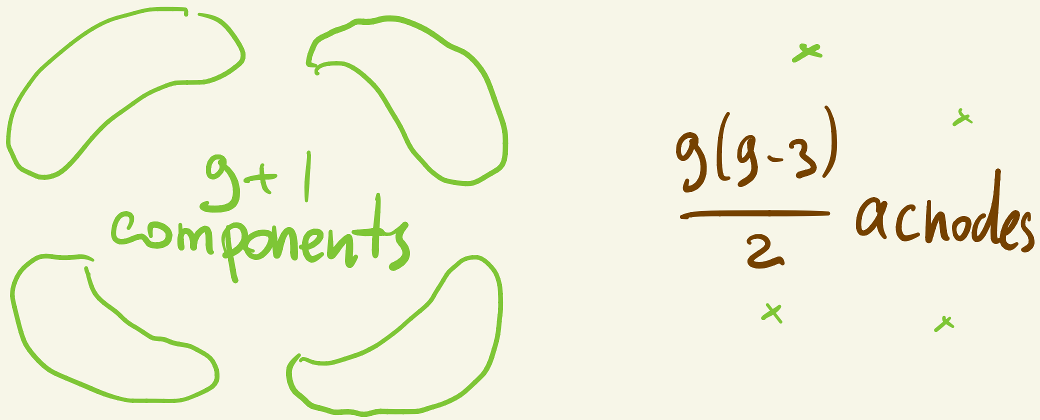
$$
\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_W \mathrm{Pic}^d C \arrow[d] \arrow[dr, dashed, "\tilde{\Lambda}"] & \\
W \arrow[r, hook] & \mathrm{Pic}^d C \arrow[r, dashed, "\Lambda"] & M_{0,n}
\end{tikzcd}
$$

$D = \tilde{\Lambda}(E)$ is a divisor on $M_{0,n}$ and its equation f contributes to the numerator of the form $\underline{A}$:

$$\boxed{\underline{A} = f^2 \underline{\tilde{A}}}$$

why can't this happen along the Huisman component Pic_H?

For a general M-curve C, a general $L \in W(\mathbb{R})$ realizes C as a Harnack curve in $\mathbb{P}^2$ of degree $d = g+1$:



g odd $\Rightarrow$ all components are ovals (separate $\mathbb{RP}^2$ into a disc and a Möbius)
g even $\Rightarrow$ g components are ovals and one is a pseudoline (generates $\pi_1(\mathbb{RP}^2)$)

$\Rightarrow L \in \mathrm{Pic}_{\emptyset}$ (g odd) or $L \in \mathrm{Pic}_{0,\dots,0,1,0,\dots,0}$ (g even)

$\Rightarrow W(\mathbb{R})$ is disjoint from Pic_H

Extension to stable curves

$$C \in \overline{M}_{g,n}$$

$$\boxed{\begin{array}{c} n = g+3 \\ d = \sum \vec{d}_i = g+1 \end{array}}$$

$\vec{d}$ is a multidegree (one for each irred component)

Def $(C, \vec{d})$ is an MHV curve of a general $L \in \mathrm{Pic}^{\vec{d}} C$ is

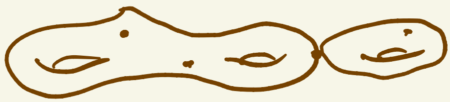
1. not special $\Leftrightarrow h^0(L) = 2$
2. globally generated
3. $\Lambda : \mathrm{Pic}^{\vec{d}} C \dashrightarrow M_{0,n}$

$$L \longmapsto \varphi_L(p_1), \ldots, \varphi_L(p_n)$$

is generically finite

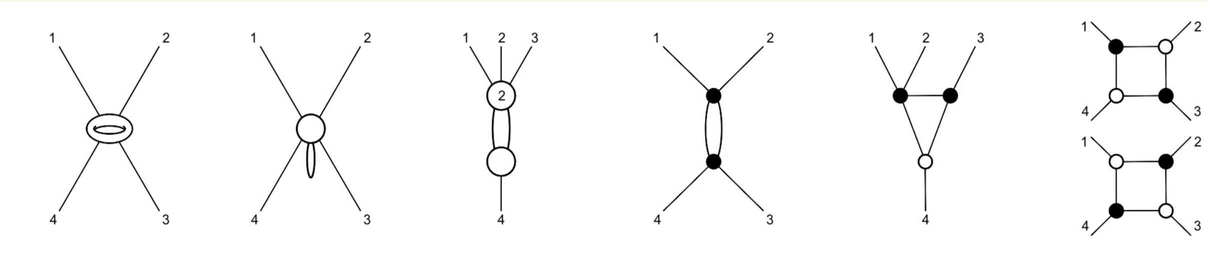
Scattering amplitude from A is a translation-invariant form on $\mathrm{Pic}^{\vec{d}} C$ viewed as a multivalued form on $M_{0,n}$ via Λ

A smooth $C \in M_{g,n}$ is always MHV but a curve with a separating node



is never MHV (one-channel factorization)

$\boxed{g=1}$ MHV curves:



On-shell diagram

$\bigcirc = \mathbb{P}^1$ (if higher genus, decorate with holes)

If ≥ 2 components, decorate with multidegrees $\bullet = 1$, $\bigcirc = \varnothing$

Theorem

Λ behaves well in families:

$$\begin{tikzcd}
\mathcal{Q}^{MHV}_{g,n,d} \arrow[r, "ev"] \arrow[d, "/GL_2"] & (\mathbb{P}^1)^n \setminus \bigcup_{i,j} D_{ij} \arrow[d, "/PGL_2"] \\
\mathcal{P}ic_0^{MHV}\mathcal{C} \arrow[r, "\Lambda"] \arrow[d] & M_{0,n} \\
\overline{M}^{MHV}_{g,n} &
\end{tikzcd}$$

open substack in the MOP stack of stable quotients $\rightarrow$ $\mathcal{Q}^{MHV}_{g,n,d}$

open substack in the stack of pairs (C,L) $\rightarrow$ $\mathcal{P}ic_0^{MHV}\mathcal{C}$

open substack of $\overline{M}_{g,n}$ $\rightarrow$ $\overline{M}^{MHV}_{g,n}$

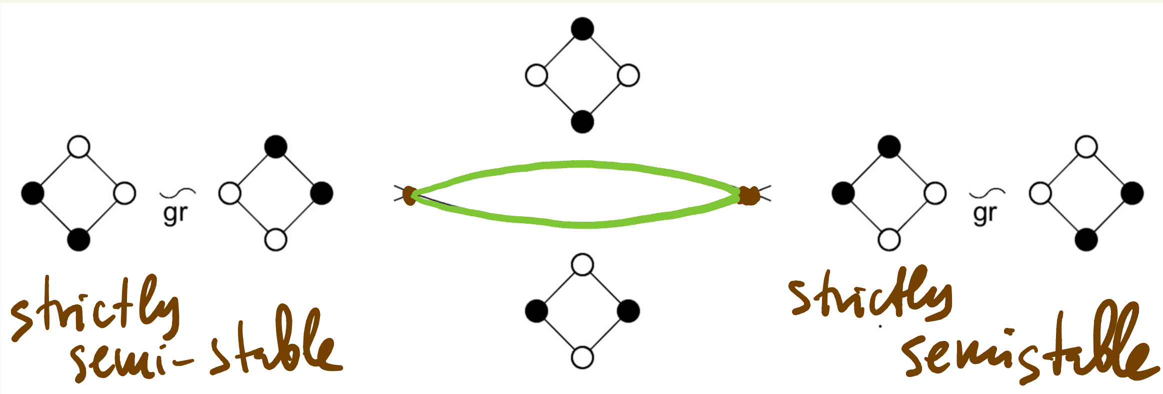
Amplification

MHV line bundles $L \in \mathcal{P}ic_0^{MHV}\mathcal{C}$ are stable with respect to the divisor $K_C + \frac{4}{n}(p_1 + \ldots + p_n)$ (which is ample on MHV curves).

$\Rightarrow$ $\mathcal{P}ic_0^{MHV}\mathcal{C}$ is separated and admits a compactification by a family of compactified Jacobians.

$\boxed{g = 1}$

$$\overline{\mathrm{Pic}^{MHV}\, \mathcal{C}}$$



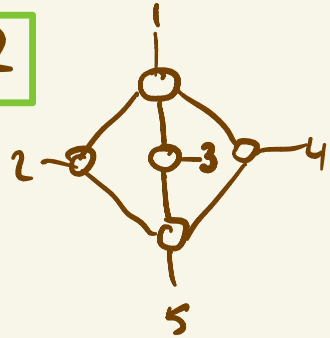
$$\Lambda \Big\downarrow 2:1$$

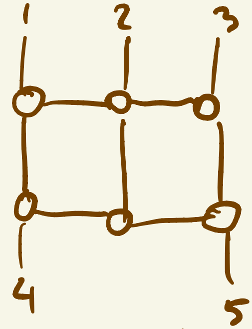
$$\overline{M}_{0,4} = \mathbb{P}^1$$

Λ is an isomorphism on each component and $A = \frac{dz}{z}$ is the same

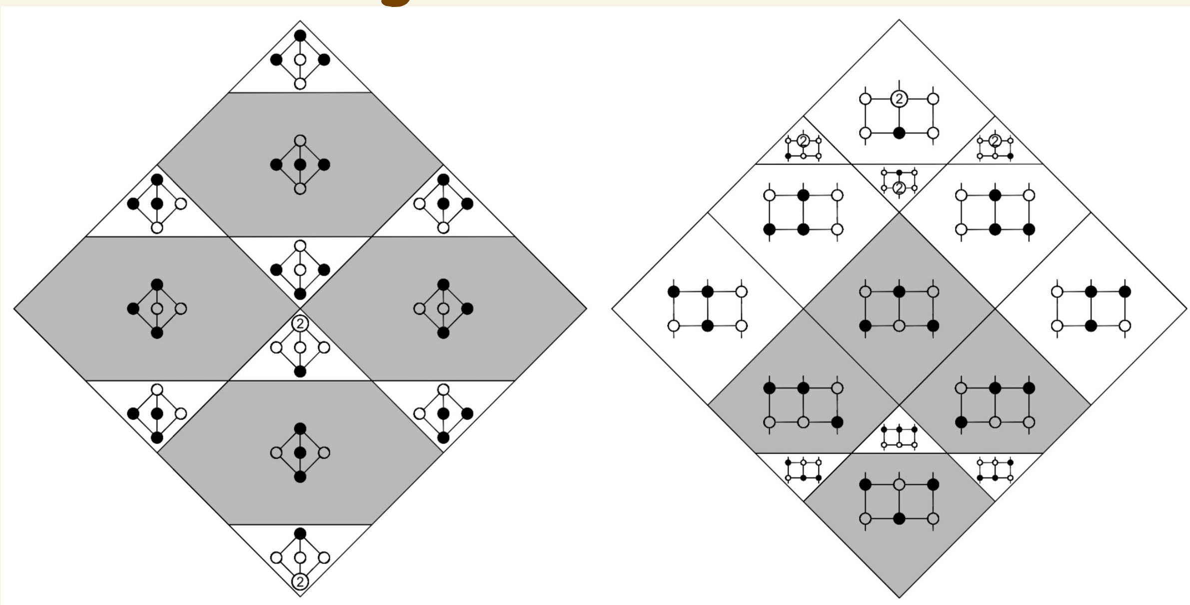
(square move)

$\boxed{g=2}$





no strictly semistable line bundles



MHV components are shaded gray

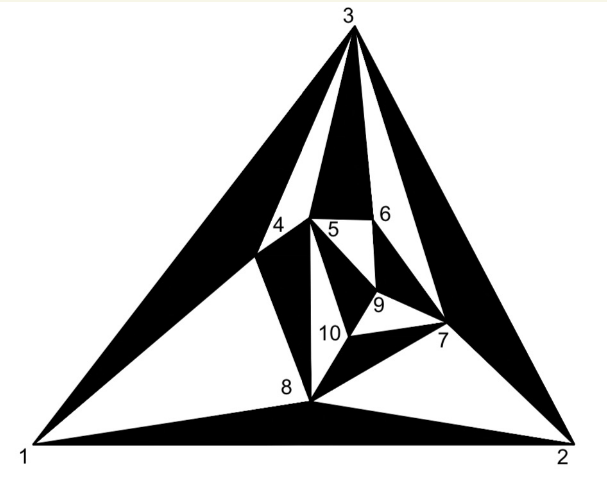
$\Lambda : \mathrm{Pic}^{MHV} C \dashrightarrow M_{0,5}$ is 4:1 with a birational restriction to each MHV component

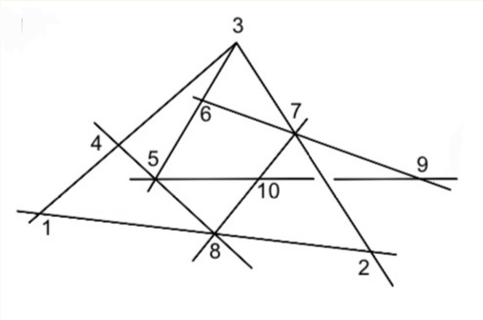
An MHV curve is maximally degenerate if every component $\cong \mathbb{P}^1$ with 3 "special" points

Lemma ([ABC+]+[Castravet-T])
Maximally degenerate MHV curves correspond to hypertrees

How about MHV M-curves?

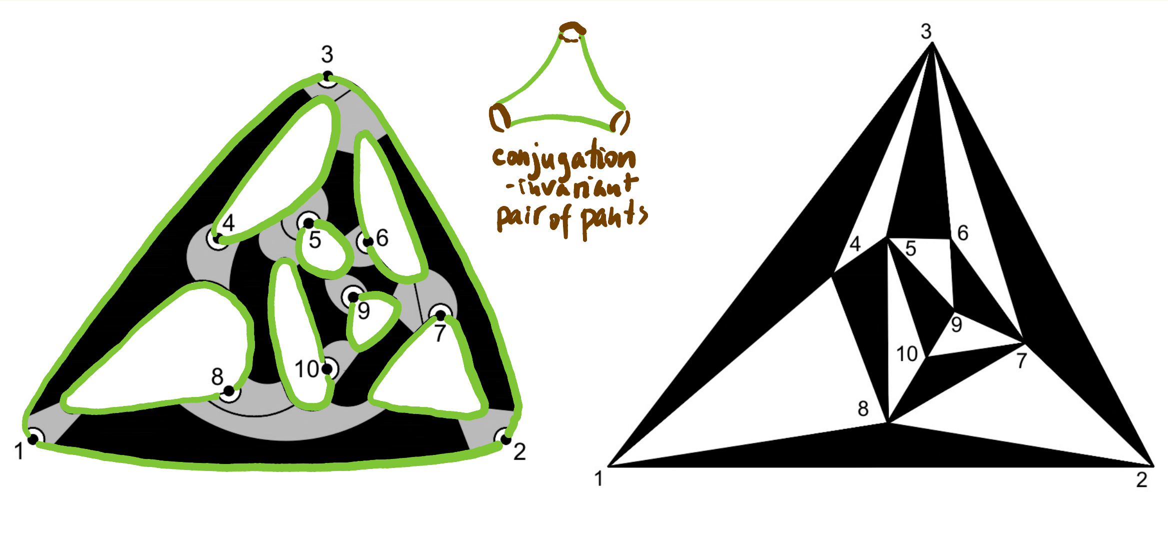
Theorem (Castravet-T) Every bicolored triangulation of S^2 gives a hypertree.





Spherical hypertree

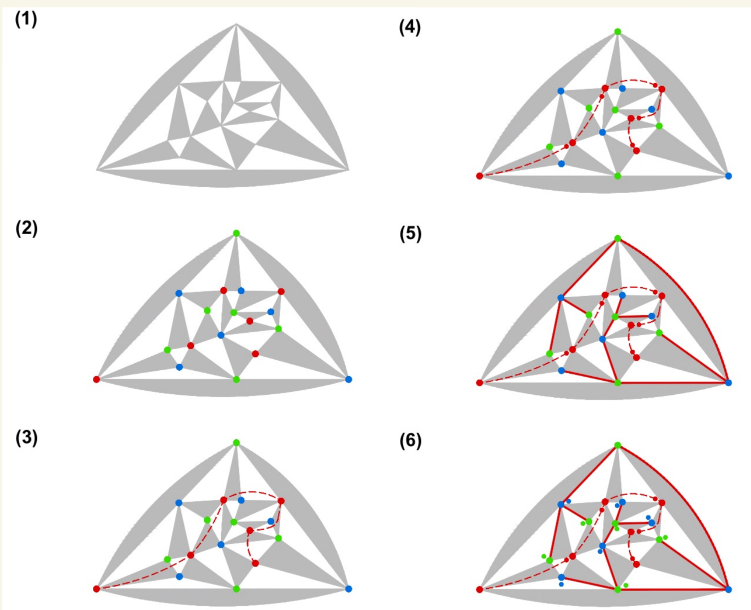
Theorem Spherical MHV curves are stable limits of real MHV M-curves of type A



Proof = results of Seppälä on stable real curves

+

Tutte Trinity Theorem



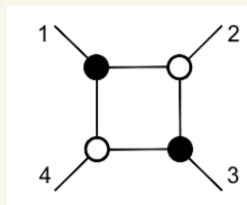
Why is our form the same as in [ABC+]?

$$\begin{tikzcd} \widetilde{\mathrm{Pic}}^d C \arrow[r] \arrow[d, "/\mathbb{G}_m^{n-1}"] & G_0(2,n) \arrow[d, "/\mathbb{G}_m^{n-1}"] \\ \mathrm{Pic}^d C \arrow[r, dashed, "\Lambda"] & M_{0,n} \end{tikzcd}$$

$$\widetilde{\mathrm{Pic}}^d C = \{ L \in \mathrm{Pic}^d C,\ L_{p_i} \overset{\varphi_i}{\simeq} \mathbb{C} \}$$

Maximally degenerate MHV case:

$$\widetilde{\mathrm{Pic}}^d C \simeq \mathbb{G}_m^{2d} \simeq \prod \widetilde{\mathrm{Pic}}^1(\ )$$



Translation-invariant form on

$$\widetilde{\mathrm{Pic}}^1(\ ) \simeq \mathbb{G}_m^3/\mathbb{G}_m \quad \text{is}$$

$$\frac{1}{\mathbb{G}_m} \frac{da}{a} \wedge \frac{db}{b} \wedge \frac{dc}{c}$$

Wedging them together gives the translation-invariant form Λ on $\widetilde{\mathrm{Pic}}^d C$

A similar $\mathbb{G}_m^d$-torsor over $G_0(2,n)$ gives a non-vanishing top degree form

$$A_0 = \prod \left(\frac{1}{\mathbb{G}_m} \frac{d(ab)}{(ab)} \wedge \frac{d(ac)}{(ac)} \wedge \frac{d(bc)}{(bc)} \right)$$

We have $A = \Psi A_0$

$\nwarrow$ Jacobian matrix

Take a $d \times (d+2)$ matrix

triple $\{a,b,c\}$
$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} \cdots (bc) \cdots (ac) \cdots (ab) \cdots \end{bmatrix} \end{array}$$

Lemma ([ABC+] for leading singularity)

$\Psi = \mathrm{Det}^2$, where Reduced determinant

$$\mathrm{Det} = \frac{1}{(ab)} \det \left[\begin{matrix} \text{remove} \\ \text{columns } a, b \end{matrix} \right]$$

(does not depend on a,b)

"Physics" ☺ (Complexified) momenta of n 4D particles $\Leftrightarrow$ 2×2 matrices $P_1 \ldots P_n$

Momentum conservation $\Leftrightarrow \sum P_i = 0$

Massless particles $\Leftrightarrow \det P_i = 0$

Write $P_i = \lambda_i \tilde{\lambda}_i^T$, spinor variables (2D)

Arrange spinor variables into $2\times n$ matrices $\lambda, \tilde{\lambda}$

[Eventually, n points in $\mathbb{P}^1$ will correspond to λ and $\tilde{\lambda}$ will be ignored]

Momentum conservation $\Leftrightarrow \lambda \cdot \tilde{\lambda}^T = 0 \Leftrightarrow \delta^{2\times 2}(\lambda \tilde{\lambda})$

Little group $(\lambda_i, \tilde{\lambda}_i) \to (t_i \lambda_i, t_i^{-1} \tilde{\lambda}_i^{-1})$

Not supersymmetric scattering amplitudes

$$A_n(t_i \lambda_i, t_i^{-1} \tilde{\lambda}_i; h_i) = \prod t_i^{-2h_i} A_n(\lambda_i, \tilde{\lambda}_i; h_i)$$

half-integer helicities

For $\mathcal{N}=4$ SYM, helicity states can be assembled into a Grassmann coherent state labelled by Grassmann variables η_i^I, $I=1,..4$ (scales like $\tilde{\lambda}_i$ under little group)

Complete superamplitude

$$A_n(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i) = \sum A_n^{(k)}(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i)$$

degree $4k$ in $\tilde{\eta}_i$.

MHV case $\Leftrightarrow k=2$

Theorem (ABC+) Leading singularity of the MHV amplitude given by an on-shell diagram is

$$\underset{\text{"out" scattering amplitude}}{\mathcal{A}} \underbrace{\delta^{2\times 4}(\lambda \cdot \tilde{\eta}^T)\,\delta^{2\times 2}(\lambda\,\tilde{\lambda}^T)}_{\text{supermomentum conservation}}$$